8.23 (a) Note that

$$\det(A-\lambda I)=0 \Longleftrightarrow \det\begin{bmatrix}3-\lambda & 0\\ 8 & -1-\lambda\end{bmatrix}=0 \Longleftrightarrow (3-\lambda)(-1-\lambda)=0.$$

So the eigenvalues are $\lambda=3$ and $\lambda=-1$.

(b) Note that

$$\det(A-\lambda I)=0 \Longleftrightarrow \det\begin{bmatrix}10-\lambda & -9\\ 4 & -2-\lambda\end{bmatrix}=0 \Longleftrightarrow (10-\lambda)(-2-\lambda)+36=0$$

$$\Longleftrightarrow \lambda^2-8\lambda+16=0 \Longleftrightarrow (\lambda-4)^2=0.$$

So $\lambda=4$ is the only eigenvalue.

(c) Note that

$$\det(A-\lambda I)=0 \Longleftrightarrow \det\begin{bmatrix}-2-\lambda & -7\\ 1 & 2-\lambda\end{bmatrix} \Longleftrightarrow (-2-\lambda)(2-\lambda)+7=0 \Longleftrightarrow \lambda^2+3=0.$$

Hence, the matrix doesn't have any eigenvalue.

8.24 Because

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & 0 & 1\\ -2 & 1-\lambda & 0\\ -2 & 0 & 1-\lambda\end{bmatrix}=(1-\lambda)\det\begin{bmatrix}4-\lambda & 1\\ -2 & 1-\lambda\end{bmatrix}$$

$$=(1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right]=(1-\lambda)\left[\lambda^2-5\lambda+6\right]$$

$$=(1-\lambda)(\lambda-2)(\lambda-3),$$

we find that $\det(A-\lambda I)=0 \Longleftrightarrow \lambda=1$ or $\lambda=2$ or $\lambda=3$.

Hence, $\lambda=1$, $\lambda=2$ and $\lambda=3$ are the eigenvalues of the matrix $A$.

8.25 (a) The eigenvalues are $\lambda=-1$ and $\lambda=5$.

(b) The eigenvalues are $\lambda=3$, $\lambda=7$ and $\lambda=1$.

(c) The eigenvalues are $\lambda=-\frac{1}{2}$, $\lambda=-\frac{1}{3}$, $\lambda=1$ and $\lambda=\frac{1}{2}$.

8.26 The eigenvalues of the matrix $A$ are $\lambda=1$, $\lambda=\frac{1}{2}$, $\lambda=0$ and $\lambda=2$.

The eigenvalues of the matrix $A^9$ are $\lambda=1$, $\lambda=\left(\frac{1}{2}\right)^9$, $\lambda=0$ and $\lambda=2^9$.

8.27 (a) For the eigenvalue $\lambda=3$, reduction of the coefficient matrix of the system $(A-\lambda I)\underline{x}=\underline{0}$ leads to

$$\begin{bmatrix}0 & 0\\ 8 & -4\end{bmatrix}\to\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & 0\end{bmatrix}.$$

Hence,

$$E_3=\left\{c\begin{bmatrix}1\\ 2\end{bmatrix} \mid c\in\mathbb{R}\right\}.$$

So the vector $\begin{bmatrix}1\\ 2\end{bmatrix}$ spans the eigenspace $E_3$.

For the eigenvalue $\lambda=-1$, reduction of the coefficient matrix of the system $(A-\lambda I)\underline{x}=\underline{0}$ leads to

$$\begin{bmatrix}4 & 0\\ 8 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

Hence,

$$E_{-1} = \left\{ c \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid c \in \mathbb{R} \right\}.$$

So the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ spans the eigenspace $E_{-1}$.

(b) For the eigenvalue $\lambda = 4$, reduction of the coefficient matrix of the system $(A - \lambda I)\underline{x} = \underline{0}$ leads to

$$\begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}.$$

Hence,

$$E_{4} = \left\{ c \begin{bmatrix} 3 \\ 2 \end{bmatrix} \mid c \in \mathbb{R} \right\}.$$

So the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ spans the eigenspace $E_4$.

(c) There are no eigenspaces.

8.32 If $A\underline{v} = \lambda\underline{v}$ and $\underline{v} \neq \underline{0}$, then

$$(A - cI)\underline{v} = A\underline{v} - cI\underline{v} = \lambda\,\underline{v} - c\,\underline{v} = (\lambda - c)\underline{v}.$$

Hence, $\underline{v}$ is an eigenvector for the matrix $A - cI$ corresponding to the eigenvalue $\lambda - c$.

8.33 Note that

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I).$$

So $\det(A - \lambda I) = 0 \iff \det(A^T - \lambda I) = 0$. The solutions of the first equation are the eigenvalues of the matrix $A$ (if any) and the solutions of the second equation are the eigenvalues of the matrix $A^T$ (if any).

9.1 We will prove that the zero vector is unique.

Suppose that $\underline{0}$ and $\underline{0}'$ are two zero vectors. Then, in view of property (3),

$$\underline{0} + \underbrace{\underline{0}'}_{\text{zero vector}} = \underline{0} \quad \text{and} \quad \underbrace{\underline{0}}_{\text{zero vector}} + \underline{0}' = \underline{0}'.$$

So $\underline{0} = \underline{0} + \underline{0}' = \underline{0}'$. Hence the two zero vectors are equal.

We will prove that each vector $\underline{u}$ in $V$ has a unique opposite.

Assume that $\underline{u}_1$ and $\underline{u}_2$ are both opposites of $\underline{u}$. Then, in view of property (4),

$$\underline{u} + \underline{u}_1 = \underline{u} + \underline{u}_2 = \underline{0}.$$

So

$$\begin{aligned} \underline{u}_1 &\overset{\text{property 3}}{=} \underline{u}_1 + \underline{0} = \underline{u}_1 + (\underline{u} + \underline{u}_2) \overset{\text{property 2}}{=} (\underline{u}_1 + \underline{u}) + \underline{u}_2 \\ &\overset{\text{property 1}}{=} (\underline{u} + \underline{u}_1) + \underline{u}_2 = \underline{0} + \underline{u}_2 \overset{\text{property 3}}{=} \underline{u}_2. \end{aligned}$$

Hence, the vectors $\underline{u}_1$ and $\underline{u}_2$ are equal.

9.2 Let $\underline{u} \in V$ and $c \in \mathbb{R}$ and assume that $c\,\underline{u} = \underline{0}$.

Suppose that $c \neq 0$ [if $c = 0$, then the proof is complete!]. Then

$$c\,\underline{u} = \underline{0} \Longrightarrow \frac{1}{c} \cdot \big(c\,\underline{u}\big) = \frac{1}{c} \cdot \underline{0}.$$

However,

$$\frac{1}{c} \cdot \big(c\,\underline{u}\big) \overset{\text{property 7}}{=} \big(\frac{1}{c} \cdot c\big)\underline{u} = 1 \cdot \underline{u} \overset{\text{property 8}}{=} \underline{u}$$

and in view of Theorem 1(b),

$$\frac{1}{c} \cdot \underline{0} = \underline{0}.$$

Hence, $\underline{u} = \underline{0}$.

9.3 Because

$$(-1)\underline{u} = \begin{bmatrix} -u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad -\underline{u} = \begin{bmatrix} -u_1 \\ -u_2 \\ -u_3 \end{bmatrix},$$

in general $-\underline{u} \neq (-1)\underline{u}$. In view of property (c) of Theorem 1 this implies that the set $V$ is not a vector space.

9.4 We will prove that the set $V$ together with the operations defined in the exercise is a vector space.

First of all note that the non-empty set $V$ is closed with respect to addition and scalar multiplication.

If $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} y \\ 0 \end{bmatrix}$ are in $V$, then $\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} x+y \\ 0 \end{bmatrix}$ is contained in $V$.

If $\begin{bmatrix} x \\ 0 \end{bmatrix} \in V$ and $c \in \mathbb{R}$, then $c\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} cx \\ 0 \end{bmatrix} \in V$.

Next we will check all the properties.

(1) According to the properties of real numbers,

$$\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} x+y \\ 0 \end{bmatrix} = \begin{bmatrix} y+x \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

(2) According to the properties of real numbers,

$$\begin{aligned} \begin{bmatrix} x \\ 0 \end{bmatrix} + \left(\begin{bmatrix} y \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} y+z \\ 0 \end{bmatrix} = \begin{bmatrix} x+(y+z) \\ 0 \end{bmatrix} = \begin{bmatrix} (x+y)+z \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} x+y \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ 0 \end{bmatrix} = \left(\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ 0 \end{bmatrix}\right) + \begin{bmatrix} z \\ 0 \end{bmatrix}. \end{aligned}$$

(3) Note that the vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$ is the zero vector because

$$\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x+0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

and, similarly, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$.

(4) Note that the vector $\begin{bmatrix} -x \\ 0 \end{bmatrix} \in V$ is the negative of the vector $\begin{bmatrix} x \\ 0 \end{bmatrix}$:

$$\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} -x \\ 0 \end{bmatrix} = \begin{bmatrix} x+(-x) \\ 0 \end{bmatrix} = \begin{bmatrix} x-x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(5) According to the properties of real numbers,

$$c\left(\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ 0 \end{bmatrix}\right) = c\begin{bmatrix} x+y \\ 0 \end{bmatrix} = \begin{bmatrix} c(x+y) \\ 0 \end{bmatrix} = \begin{bmatrix} cx+cy \\ 0 \end{bmatrix} = \begin{bmatrix} cx \\ 0 \end{bmatrix} + \begin{bmatrix} cy \\ 0 \end{bmatrix} = c\begin{bmatrix} x \\ 0 \end{bmatrix} + c\begin{bmatrix} y \\ 0 \end{bmatrix}.$$

(6) According to the properties of real numbers,

$$(c+d)\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} (c+d)x \\ 0 \end{bmatrix} = \begin{bmatrix} cx+dx \\ 0 \end{bmatrix} = \begin{bmatrix} cx \\ 0 \end{bmatrix} + \begin{bmatrix} dx \\ 0 \end{bmatrix} = c\begin{bmatrix} x \\ 0 \end{bmatrix} + d\begin{bmatrix} x \\ 0 \end{bmatrix}.$$

(7) According to the properties of real numbers,

$$c\left(d\begin{bmatrix} x \\ 0 \end{bmatrix}\right) = c\begin{bmatrix} dx \\ 0 \end{bmatrix} = \begin{bmatrix} c(dx) \\ 0 \end{bmatrix} = \begin{bmatrix} (cd)x \\ 0 \end{bmatrix} = (cd)\begin{bmatrix} x \\ 0 \end{bmatrix}.$$

(8) According to the properties of real numbers,

$$1 \cdot \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Note that the fact that $V$ is a subset of $\mathbb{R}^2$ implies that it is sufficient to check the properties 3 and 4. Since the other properties are satisfied for the elements of $\mathbb{R}^2$, they are certainly satisfied for the elements of $V$.

9.7 Let $A, B \in \mathbb{M}_{2\times 2}$ and $k \in \mathbb{R}$. Then

$$T(A+B) = \begin{bmatrix} a_{11}+b_{11} \\ \vdots \\ a_{22}+b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} \\ \vdots \\ b_{22} \end{bmatrix} = T(A) + T(B)$$

and

$$T(kA) = \begin{bmatrix} ka_{11} \\ \vdots \\ ka_{22} \end{bmatrix} = k\begin{bmatrix} a_{11} \\ \vdots \\ a_{22} \end{bmatrix} = k \cdot T(A).$$

So the mapping $T$ is linear.

The mapping $T$ is one-to-one:

If $T(A) = T(B)$, then

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} \Longrightarrow a_{11} = b_{11}, a_{12} = b_{12}, a_{21} = b_{21} \text{ and } a_{22} = b_{22} \Longrightarrow A = B.$$

The mapping $T$ is surjective:

If $\underline{v} \in \mathbb{R}^4$, then

$$T\left(\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}\right) = \underline{v}.$$

9.9 (a) Let $p, q \in \mathbb{P}_2$ and $k \in \mathbb{R}$. Say $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$, where $x \in \mathbb{R}$. Note that

$$(p+q)(x) = a_0 + b_0 + (a_1+b_1)x + (a_2+b_2)x^2.$$

So for the polynomial $T(p+q)$ it holds that for all $x \in \mathbb{R}$

$$\begin{aligned} T(p+q)(x) &= a_0 + b_0 + (a_1 + b_1)(x+1) + (a_2 + b_2)(x+1)^2 \\ &= a_0 + a_1(x+1) + a_2(x+1)^2 + b_0 + b_1(x+1) + b_2(x+1)^2 \\ &= T(p)(x) + T(q)(x), \end{aligned}$$

or: $T(p+q) = T(p) + T(q)$.

Observe that

$$(k \cdot p)(x) = ka_0 + ka_1 x^2 + ka_2 x^2.$$

So for the polynomial $T(k \cdot p)$ it holds that for all $x \in \mathbb{R}$

$$T(k \cdot p)(x) = ka_0 + ka_1(x+1) + ka_2(x+1)^2 = k[a_0 + a_1(x+1) + a_2(x+1)^2] = k \cdot T(p),$$

or: $T(k \cdot p) = kT(p)$. So $T$ is a linear mapping .