8.22 (a) For any $k = 0, 1, 2, \dots$

$$M\underline{v}_k = M\big[\lambda^k \underline{u}\big] = \lambda^k\, M\underline{u} = \lambda^k \cdot \lambda\underline{u} = \lambda^{k+1}\underline{u} = \underline{v}_{k+1}.$$

(b) For any $k = 0, 1, 2, \dots$

$$M\underline{w}_k = M\big[c\,\lambda^k \underline{u} + d\,\mu^k \underline{v}\big] = c\,\lambda^k\, M\underline{u} + d\,\mu^k\, M\underline{v} = c\,\lambda^k \cdot \lambda\,\underline{u} + c\,\mu^k \cdot \mu\,\underline{v} = c\,\lambda^{k+1}\underline{u} + d\,\mu^{k+1}\underline{v} = \underline{w}_{k+1}.$$

8.28 We reduce, for the eigenvalue $\lambda = 1$, the matrix $A - \lambda I$:

$$A - I = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So for any solution of the system $(A - I)\underline{x} = \underline{0}$ it holds that $x_1 = x_3 = 0$. Hence, the vector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

spans the eigenspace $\mathbb{E}_1$.

We reduce, for the eigenvalue $\lambda = 2$, the matrix $A - \lambda I$:

$$A - 2I = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So for any solution of the system $(A - 2I)\underline{x} = \underline{0}$ it holds that $\begin{cases} 2x_1 = -x_3 \\ x_2 = x_3 \end{cases}$. Hence, the vector

$$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

spans the eigenspace $\mathbb{E}_2$.

We reduce, for the eigenvalue $\lambda = 3$, the matrix $A - \lambda I$:

$$A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So for any solution of the system $(A - 3I)\underline{x} = \underline{0}$ it holds that $\begin{cases} x_1 = -x_3 \\ x_2 = x_3 \end{cases}$. Hence, the vector

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

spans the eigenspace $\mathbb{E}_3$.

8.30 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\det(A - \lambda I) = 0 \Longleftrightarrow \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = 0 \Longleftrightarrow (a - \lambda)(d - \lambda) - bc = 0$$
$$\Longleftrightarrow \lambda^2 - (a + d)\lambda + ad - bc = 0 \Longleftrightarrow \lambda^2 - \mathrm{tr}(A)\lambda + \det A = 0.$$

8.31 Let λ be an eigenvalue of an invertible matrix A.

If $\lambda = 0$ and $\underline{v}$ is an eigenvector corresponding to this eigenvalue, then $\underline{v} \neq \underline{0}$ and $A\underline{v} = \lambda\,\underline{v} = \underline{0}$. This is in contradiction with the invertibility of the matrix A. Hence $\lambda \neq 0$.

We have

$$A\underline{v} = \lambda\underline{v} \Longrightarrow A^{-1}A\underline{v} = A^{-1}\lambda\underline{v} \Longrightarrow \underline{v} = \lambda A^{-1}\underline{v} \Longrightarrow A^{-1}\underline{v} = \frac{1}{\lambda}\underline{v}.$$

So $\dfrac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1}.

8.34 Note that the entries in each row of the matrix A^T sum up to 1. Hence,

$$A^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

which proves that $\lambda = 1$ is an eigenvalue of the matrix A^T. Due to Exercise 33, $\lambda = 1$ is also an eigenvalue of the matrix A.

8.35 (a) Note that

$$A^T = [\lambda\,\underline{u}\,\underline{u}^T + \mu\,\underline{v}\,\underline{v}^T]^T = \lambda[\underline{u}\,\underline{u}^T]^T + \mu[\underline{v}\,\underline{v}^T]^T = \lambda\big(\underline{u}^T\big)^T\underline{u}^T + \mu\big(\underline{v}^T\big)^T\underline{v}^T$$
$$= \lambda\,\underline{u}\,\underline{u}^T + \mu\,\underline{v}\,\underline{v}^T = A.$$

(b) Note that for any $\underline{x} \in \mathbb{R}^n$

$$A\underline{x} = \lambda\,\underline{u}\,\underline{u}^T\underline{x} + \mu\,\underline{v}\,\underline{v}^T\underline{x} = \lambda\,(\underline{u} \cdot \underline{x})\,\underline{u} + \mu\,(\underline{v} \cdot \underline{x})\,\underline{v}.$$

Because $\underline{u} \cdot \underline{u} = 1$ and $\underline{u} \cdot \underline{v} = 0$, it follows that

$$A\underline{u} = \lambda\,(\underline{u} \cdot \underline{u})\,\underline{u} + \mu\,(\underline{v} \cdot \underline{u})\,\underline{v} = \lambda\,\underline{u}.$$

Since $\underline{u}$ has length one, $\underline{u} \neq \underline{0}$. Therefore $\underline{u}$ is an eigenvector of the matrix A corresponding to the eigenvalue λ.

9.5 Note that W is not closed with respect to scalar multiplication: the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is contained in W but the matrix

$$2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

is not an element of W.

9.6 We will show that the set

$$W = \{f \in \mathbb{F} |\ f(1) = 0\}$$

is a vector space.

Note that the set W is not empty because it contains the null function n defined by $n(x) = 0$ for all $x \in \mathbb{R}$.

First of all we will show that the set W is closed under addition and scalar multiplication: if $f \in W$, $g \in W$ and $c \in \mathbb{R}$, then $f + g \in W$ and $cf \in W$.

Let $f \in W$, $g \in W$ and $c \in \mathbb{R}$. Then $f, g \in \mathbb{F}$, $f(1) = 0$ and $g(1) = 0$.

So $f + g \in \mathbb{F}$, $cf \in \mathbb{F}$ and

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$$

and
$$(cf)(1) = c \cdot f(1) = c \cdot 0 = 0.$$

Hence, $f + g \in W$ and $cf \in W$.

Next we have to check whether the properties (1) up to (8) are satisfied!

Because $\mathbb{F}$ is a vector space, as we may conclude from Example 3, the properties (1), (2), (5), (6), (7) and (8) hold for the space W as well.

Furthermore, since the null function n is an element of the set W, property (3) is also satisfied.

If f is an element of the set W, then $-f \in \mathbb{F}$ and $(-f)(1) = -f(1) = 0$. By consequence, $-f \in W$.

Property (4) is also satisfied.

9.8 (a) Let $A, B \in \mathbb{M}_{n\times n}$ and $k \in \mathbb{R}$. Then

$$T(A + B) = \text{tr}(A + B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \text{tr}(A) + \text{tr}(B) = T(A) + T(B)$$

and
$$T(kA) = \text{tr}(kA) = \sum_{i=1}^{n}(ka_{ii}) = k\sum_{i=1}^{n} a_{ii} = k\,\text{tr}(A) = k\,T(A).$$

So the mapping T is linear.

The mapping T is not one-to-one:

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right).$$

The mapping T is surjective:

If $c \in \mathbb{R}$, then $T\left(\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\right) = c.$

(b) Let $A, B \in \mathbb{M}_{m\times n}$ and $k \in \mathbb{R}$. Then

$$T(A + B) = (A + B)^T = A^T + B^T = T(A) + T(B)$$

and
$$T(kA) = (kA)^T = kA^T = kT(A).$$

So the mapping T is linear.

The mapping T is one-to-one:

If $A, B \in \mathbb{M}_{m\times n}$ and $T(A) = T(B)$, then $A^T = B^T$. Hence

$$A = \left(A^T\right)^T = \left(B^T\right)^T = B.$$

The mapping T is surjective:

If $C \in \mathbb{M}_{n\times m}$, then $C^T \in \mathbb{M}_{m\times n}$ and

$$T(C^T) = \left(C^T\right)^T = C.$$